A Kovacic-style algorithm for liouvillian solutions of third order differential equations

Michael F. Singer*
North Carolina State University
Department of Mathematics
Raleigh, N.C. 27695-8205
singer@math.ncsu.edu

Felix Ulmer
IRMAR, Université de Rennes 1
Campus de Beaulieu
F-35042 Rennes Cedex
ulmer@univ-rennes1.fr

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Abstract

The Kovacic algorithm [7] and its improvements (e.g. [18]) give explicit formulas for the Liouvillian solutions of second order linear differential equations. In this paper we extend this approach to third order equations. The result will either be a formula or a proof that the differential Galois group is one of 12 finite groups (it is well known how to solve a differential equation with known finite differential Galois group [3, 14, 17]). The approach uses only linear algebra, factorization of differential operators and computation of exponential solutions (all available in Maple), making the result easier to understand and to use for non specialists than [6]. The algorithm also decides if a third order differential equation which cannot be solved in terms of Liouvillian solutions can be solved using solutions of second order equations.
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1 Introduction and Notations

In this paper k is a differential field whose field of constants $\mathcal{C}$ is algebraically closed of characteristic 0 (e.g. $\overline{\mathbb{Q}}(x)$ with the usual derivation d/dx). For the derivation δ of k and $a \in k$ we write $\delta^k(a) = a^{(k)}$ and also $a^{(1)} = a'$, $a^{(2)} = a''$, ... Let

$$L(y) = a_n\ y^{(n)} + a_{n-2}\ y^{(n-2)} + \ldots + a_1\ y' + a_0\ y \ = \ 0, \qquad a_i \in k \qquad (1)$$

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be a linear differential equation of order n over k. Using the variable transformation $y = z \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$, it is always possible to transform a given differential equation in the above form without loosing the Liouvillian character of the solutions. We denote by $\{y_1, \ldots, y_n\}$ a fundamental set of solutions of (1) and we refer to [12, 14] for precise definitions of a *Picard-Vessiot extension* (PVE) of k , *differential Galois group* $\mathcal{G}$ and *liouvillian solutions* for (1). A PVE K of k for (1) is a differential field extension $K = k < y_1, \ldots, y_n >$ with no new constants which is the equivalent of a splitting field for (1). Under our assumptions a PVE exists and is unique up to differential automorphisms. The group $\mathcal{G}$, which consists of the differential automorphisms of K/k, is a linear algebraic group over $\mathcal{C}$ which sends a solution of (1) into a solution of (1). This action gives a faithful representation of $\mathcal{G}$ as a subgroup of $SL(n, \mathcal{C})$ which will be *the* representation of $\mathcal{G}$ in what follows. There is a Galois correspondence between algebraic subgroups of $\mathcal{G}$ and differential subfields of the PVE of (1) and the fixed field of $\mathcal{G}$ is k. A solution of (1) in k is a *rational solution*, a solution in an algebraic extension of k is an *algebraic solution*, a solution whose derivative is in k is an *integral* and a solution whose logarithmic derivative is in k is an *exponential solution* and a solution belonging to a field obtained by successive adjunctions of exponentials, integrals and algebraic functions is a *liouvillian solution*.

A linear differential equation $L(y) = 0$ is reducible (as differential operator) if there exists a decomposition $L(y) = L_1(L_2(y))$ in operators of lower order. Algorithms to factor differential operators exist for $k = \overline{\mathbb{Q}}(x)$ and are implemented in Maple [5]. The equation $L(y) = 0$ is reducible if and only if $\mathcal{G} \subseteq SL(n, \mathcal{C})$ is a reducible linear group.

Definition 1.1 Let $G \subset GL(V)$ be a linear group acting irreducibly on the vector space V of dimension n over $\mathcal{C}$. Then G is called *imprimitive* if, for $k > 1$, exist subspaces $V_1, \cdots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$ and, for each $g \in G$, the mapping $V_i \to g(V_i)$ is a permutation of the set $\mathcal{S} = \{V_1, \ldots, V_k\}$. The set $\mathcal{S}$ is called a *system of imprimitivity* of G. If all the subspaces V_i are one dimensional, then G is called *monomial*. An irreducible group $G \subseteq GL(V)$ which is not imprimitive is called *primitive*.

We note that, since an imprimitive group G is assumed to act irreducibly on V, we have that G acts transitively on the V_i. In particular, all the V_i have the same dimension.

Definition 1.2 *Let V be a $\mathcal{C}$-vector space, let $\{Y_1, \ldots, Y_n\}$ be a basis for V, and let $G \subseteq GL(V)$ be a linear group. Define a group action of $g \in G$ on $f \in \mathcal{C}[Y_1, \ldots, Y_n]$ by $g \cdot (f(Y_1, \ldots, Y_n)) = f(g(Y_1), \ldots, g(Y_n))$. A homogeneous polynomial of degree m with the property that*

$$\forall g \in G, \quad g \cdot f(Y_1, \ldots, Y_n) = \psi_f(g) \cdot (f(Y_1, \ldots, Y_n)), \text{ with } \psi_f(g) \in \mathcal{C}$$

is called a semi-invariant *of degree m of G. The function $\psi_f(g): G \to \mathcal{C}$ is a one dimensional character of G. The polynomial f is an invariant if and only if ψ_f is the trivial character $\mathbb{1}$ of G. If there exists an integer m such that ψ_f^m is the trival character, then we call the smallest such integer the order of the character ψ_f and of the semi-invariant f.*

For an equation (1) we denote $\{y_1, \ldots, y_n\}$ a fundamental set of solutions and $(Y_1, \ldots, Y_n)$ a basis of $V = \mathcal{C}^n$. If we now consider the evaluation morphism

$$\begin{array}{rcl} \varphi: C[Y_1, \ldots Y_n] & \to & K \\ Y_i & \mapsto & y_i \end{array}$$

then we have the following properties:

1. Linear forms are sent to solutions of $L(y) = 0$.

2. Semi-invariants of finite order m are send to m-th roots of elements of k (in particular invariants are fixed under the action of $\mathcal{G}$ and are send to elements of k)

The following theorem shows that the existence of Liouvillian solutions is equivalent to the existence of semi-invariants that factors into linear forms:

Theorem 1.3 *[16] Let $L(y) = 0$ be an n-th order linear differential equation over k with differential Galois group $\mathcal{G} \subseteq GL(n, \mathcal{C})$.*

1. *If $L(y) = 0$ has a liouvillian solution z whose logarithmic derivative $u = z'/z$ is algebraic of degree m, then there is a $\mathcal{G}(L)$-semi-invariant I of degree m in $\mathcal{C}[Y_1, \ldots, Y_n]$ which factors into linear forms.*

2. *If there is a $\mathcal{G}(L)$-semi-invariant I of degree m in $\mathcal{C}[Y_1, \ldots, Y_n]$ which factors into linear forms, then $L(y) = 0$ has a liouvillian solution z whose logarithmic derivative $u = z'/z$ is algebraic of degree $\leq m$.*

Our goal will be to compute the minimal polynomial $P(U)$ of $u = z'/z$, i.e. to compute a solution z of the form $e^{\int u}$. Note that each zero u_i of $P(U)$ is a conjugate of u under $\mathcal{G}$ and thus is also the logarithmic derivative of a solution $z_i = e^{\int u_i}$ of $L(y) = 0$. In particular the minimal polynomial $P(U) = \sum_i^m f_{m-i} U^i \in k[U]$ of u is of the form

$$\left(\prod_{i=1}^m z_i\right)\left(U - \frac{z_i'}{z_i}\right)\left(U - \frac{z_i'}{z_i}\right)\dots\left(U - \frac{z_i'}{z_i}\right) =$$

$$\left(\prod_{i=1}^m z_i\right) U^m + \left(z_1' z_2 \dots z_m + \dots z_1 \dots z_{m-1} z_m'\right) U^{m-1} + \dots + \left(z_1' z_2' \dots z_m'\right)$$

The above polynomial will in general have coefficients in k only after division by the leading coefficient $f_0 = \prod_{i=1}^m z_i$. In the following we will compute the other coefficients in the above form

$$f_i = \sum_{0 \le j_1 < \ldots < j_i \le m} z_{j_1}' \dots z_{j_s}' \frac{\prod_i z_i}{z_{j_1} z_{j_i}}$$

From [16] we get that $\prod_{i=1}^m z_i$ is the image $\varphi(I)$ of the semi-invariant that factors into linear forms. The degree of the semi-invariant I can be bounded in terms of the order of $L(y) = 0$.

Theorem 1.4 *(cf. [7, 14, 16]) Let $L(y) = 0$ be a linear differential equation of order n of the form (1). There exists a function $F(n)$ depending only of n such that if $L(y) = 0$ has a Liouvillian solution, then there exists a solution z such that $u = z'/z$ is algebraic of degree m, where $m \le F(n)$. In particular:*

1. *If $L(y)$ is an irreducible second order equation then a minimal choice for m is*
   - (a) *2 if $\mathcal{G} \subset SL(2, C)$ is an imprimitive group.*
   - (b) *4, 6 or 12 if $\mathcal{G}$ is isomorphic to one of the 3 finite primitive subgroups of $SL(2, C)$.*
2. *If $L(y)$ is an irreducible third order equation then a possible choice for m is*
   - (a) *3 if $\mathcal{G} \subset SL(3, C)$ is an imprimitive group.*
   - (b) *6, 9, 21 or 36 if $\mathcal{G}$ is isomorphic to one of the 8 finite primitive subgroups of $SL(3, C)$.*

To compute $P(U)$ one has to try the possible numbers m by increasing order. If no minimal polynomial $P(U)$ can be found, then $L(y) = 0$ has no Liouvillian solution.

2 Formulas for $P(u)$

Using (1) we define the differential polynomial ring $R = k < Z_1, \ldots, Z_m >$ where the derivation Δ is defined by

1. $\Delta(f) = \delta(f)$ for $f \in k$,
2. $\Delta(Z_i^{(s)}) = Z_i^{(s+1)}$ if $s < n-1$,
3. $\Delta(Z_i^{(n-1)}) = -\left(a_{n-2}\ Z_i^{(n-2)} + \ldots + a_1\ Z_i^{1)} + a_0\ Z_i^{(0)}\right)$

Note that the ring R contains $n \cdot m$ indeterminates $Z_i^{(j)}$ where $1 \leq i \leq m$ and $0 \leq j \leq n-1$. For integers $n \geq d_1 \geq d_2 \geq \cdots \geq d_m \geq 0$ we introduce the notation $W_{d_1,\ldots,d_m}$ for the primitive part of

$$\sum_{\sigma \in S_m} Z_{\sigma(1)}{}^{(d_1)} Z_{\sigma(2)}{}^{(d_2)} \cdots Z_{\sigma(m)}{}^{(d_m)} \quad .$$

This is the same as taking the sum over the orbit of $Z_1^{(d_1)} Z_2^{(d_2)} \ldots Z_m^{(d_1)}$ under the action of S_m on the indices and then dividing out by the coefficient of this term. If we start with the monomial $S = Z_1 Z_2 \cdots Z_m$ and take derivatives, then the resulting expressions will all be symmetric in the m indices and thus can be expressed as sums over k of the $N = \binom{n+m-1}{n-1}$ expressions $W_{d_1,\ldots,d_m}$.

$$\begin{aligned}
S &= Z_1 Z_2 \cdots Z_m \\
&= W_{0,0,\ldots,0} \\
S' &= Z_1^{(1)} Z_2 \ldots Z_m + Z_1 Z_2^{(1)} \ldots Z_m + \ldots + Z_1 \ldots Z_{m-1} Z_m^{(1)} \\
&= W_{1,0,\ldots,0} \\
S'' &= \sum_{1 \leq j \leq m} Z_j^{(2)} \frac{\prod_i Z_i}{Z_j} + \sum_{1 \leq j_1 < j_2 \leq m} Z_{j_1}^{(1)} Z_{j_2}^{(1)} \frac{\prod_i Z_i}{Z_{j_1} Z_{j_2}} \\
&= W_{2,0,\ldots,0} + W_{1,1,0,\ldots,0} \\
\ldots &= \ldots
\end{aligned}$$

Considering j derivatives of S we get a linear system of the form

$$\begin{pmatrix} S \\ S' \\ \dots \\ S^{(j)} \end{pmatrix} = A_j \begin{pmatrix} W_{0,\dots,0} \\ W_{1,0,\dots,0} \\ \dots \\ W_{n-1,n-1,\dots,n-1} \end{pmatrix}$$

If we take the first $N = \binom{n+m-1}{n-1}$ derivatives of S, then we will have more equations than there are unknowns on the right side, so the equations will be linearly dependent over k, giving a linear differential equation of degree N for S. The linear combination of the $S^{(i)}$ involving the lowest possible order of derivation is called the m-th "symmetric power" of $L(y)$ and is denoted $L^{\circledS m}(y)$. In other words if for the first time A_j has row rank less than $j+1$. i.e. a non zero element $(b_j, b_{j-1}, \dots, b_0) \in k^{j+1}$ exists such as

$$\begin{pmatrix} b_0, & b_1, & \dots, & b_j \end{pmatrix} \begin{pmatrix} S \\ S' \\ \dots \\ S^{(j)} \end{pmatrix} = \begin{pmatrix} b_0, & b_1, & \dots, & b_j \end{pmatrix} A_j = 0$$

then $L^{\circledS m}(y) = \sum_{i=0}^{j} b_j y^{(j)}$.

In particular S will be a solution of $L^{\circledS m}(y)$ (cf. [9, 10]). The key point is that if we have m solutions $z_1, z_2, \dots, z_m$ of $L(y) = 0$, then we can consider the differential subsitution homomorphism

$$\varphi: k < Z_1, \dots Z_m > \;\to\; K \tag{2}$$

$$Z_i \;\mapsto\; z_i \tag{3}$$

which is the identity on k. Clearly any product of m solutions of $L(y) = 0$ will be a solution of $L^{\circledS m}(y) = 0$ and in fact its solution space is spanned by solutions of this form (cf. [9, 10]).

Recall that we are interested in finding the coefficients of

$$\left(\prod_{i=1}^{m} z_i\right) u^m + \left(z_1' z_2 \dots z_m + \dots z_1 \dots z_{m-1} z_m'\right) u^{m-1} + \dots + \left(z_1' z_2' \dots z_m'\right)$$

which are just the images

$$w_{0,\dots,0},\ w_{1,0,\dots,0},\ w_{1,1,0,\dots,0}, \dots,\ w_{1,1,\dots,1}$$

of

$$W_{0,\dots,0},\ W_{1,0,\dots,0},\ W_{1,1,0,\dots,0}, \dots,\ W_{1,1,\dots,1}$$

under the above differential evaluation morphism. This shows that $s = z_1 z_2 \cdots z_m$ (the value of the semi-invariant that factors into linear forms) is an exponential solution of $L^{\circledS m}(y) = 0$. The case where $L^{\circledS m}(y)$ is of maximal order $N = \binom{n+m-1}{n-1}$ is particulary interesting. In this case the matrix A_{N-1} corresponding to

$$\begin{pmatrix} s \\ s' \\ \cdots \\ s^{N-1} \end{pmatrix} = A_{N-1} \begin{pmatrix} w_{0,\ldots,0} \\ w_{1,0,\ldots,0} \\ \cdots \\ w_{n-1,n-1,\ldots,n-1} \end{pmatrix}$$

will be invertible, allowing us to express the values

$$w_{0,\ldots,0},\ w_{1,0,\ldots,0},\ w_{1,1,0,\ldots,0}, \ldots,\ w_{1,1,\ldots,1}$$

of the coefficents of $P(u)$ in terms of the value s and its derivatives. We give an explicit example in the next section.

3 Second order equations

We consider a differential equation $L(y) = y'' + a_0 y = 0$, our goal is to derive the well known Kovacic algorithm using only linear algebra.

In the imprimitive case Theorem 1.4 shows that that there exists a polynomial $P(U)$ of degree $m = 2$. We thus consider

$$\begin{array}{lcllr} S & = & Z_1 Z_2 & & \\ S' & = & & Z_1' Z_2 + Z_1 Z_2' & \\ S'' & = & -2a_0\, Z_1 Z_2 & & +2\, Z_1' Z_2' \\ S''' & = & -2a_0'\, Z_1 Z_2 & -4a_0 (Z_1' Z_2 + Z_1 Z_2') & \end{array}$$

which is equivalent to the system

$$\begin{pmatrix} S \\ S' \\ S'' \\ S''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \\ -2a_0' & -4a_0 & 0 \end{pmatrix} \begin{pmatrix} W_{0,0} \\ W_{1,0} \\ W_{1,1} \end{pmatrix} = A_3 \begin{pmatrix} W_{0,0} \\ W_{1,0} \\ W_{1,1} \end{pmatrix}$$

From the matrix form we see that in this case the symmetric power $L^{\circledS 2}(y)$ will always be of maximal order 3. To compute $L^{\circledS 2}(y)$ we look for an element (b_0, b_1, b_2, b_3) in the kernel of A_3^t.

$$\left(\begin{array}{cccc} 2a_0', & 4a_0, & 0, & 1 \end{array} \right) \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \\ -2a_0' & -4a_0 & 0 \end{array} \right) = \left(\begin{array}{cccc} 0, & 0, & 0, & 0 \end{array} \right)$$

which gives us the following formula:

$$L^{\circledS 2}(y) = 2a_0' y + 4a_0 y' + y'''$$

In the case of second order equations any exponential solution s of some $L^{\circledS m}(y)$ is the value of a semi-invariant that factor into linear forms and thus lead to a polynomial $P(U)$ (cf. [18]). Thus for any exponential solution s of $L^{\circledS m}(y)$ we get

$$P(U) = w_{0,0} U^2 + w_{1,0} U + w_{1,1}.$$

By solving the linear system

$$\left(\begin{array}{c} s \\ s' \\ s'' \end{array} \right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \end{array} \right) \left(\begin{array}{c} w_{0,0} \\ w_{1,0} \\ w_{1,1} \end{array} \right)$$

This gives

$$P(U) = s\, U^2 \; + \; s'\; U + \left(\; a_0 s + \frac{s''}{2} \right)$$

This is what we mean by "formula". Note that the formula is in fact (up to smoe linear algebra) the above matrix.

Example 3.1 *Consider the irreducible equation*

$$L(y) = y'' - \frac{2}{2x-1} y' + \frac{(27x^4 - 54x^3 + 5x^2 + 22x + 27)\,(2x-1)^2}{144x^2(x-1)^2\,(x^2-x-1)^2} y = 0.$$

we first transform it into the form $y'' + a_0 y$ using the transformation

$$y = z \; e^{-\frac{1}{2}\int \frac{2}{2x-1}}$$

to get

$$y'' + \frac{1}{144} \frac{728\, x^6 - 2184\, x^5 + 2035\, x^4 - 430\, x^3 + 45\, x^2 - 194\, x + 27}{x^2\,(x-1)^2\,(2\,x-1)^2\,(x^2-x-1)^2} \; y \;\; = \;\; 0$$

Using the above formula for $L^{\circledS 2}(y)$ we get:

$$\frac{2912x^{10}-14560x^9+28590x^8-27000x^7+13445x^6-6987x^5+5450x^4-1635x^3-420x^2+205x-27}{-36x^3(x-1)^3(2x-1)^3(x^2-x-1)^3}\,y$$

$$+\frac{728x^6-2184x^5+2035x^4-430x^3+45x^2-194x+27}{36x^2(x-1)^2(2x-1)^2(x^2-x-1)^2}\,y'+y'''$$

We now compute the exponential solutions of $L^{\circledS 2}(y)$ (for example using the commmand expsols *in Maple) and get:*

$$S=e^{1/2\,\frac{11\,x^2-3\,x-1-16\,x^3+8\,x^4}{(x^2-x-1)(x-1)x(2\,x-1)}}=\frac{\sqrt{x-1}\sqrt{x}\,(x^2-x-1)}{2\,x-1}$$

By using the formula above we get for $P(U)$:

$$\frac{\sqrt{x-1}\sqrt{x}\,(x^2-x-1)}{2\,x-1}\,U^2+1/2\,\frac{11\,x^2-3\,x-1-16\,x^3+8\,x^4}{\sqrt{x-1}\,(2\,x-1)^2\sqrt{x}}\,U$$

$$+\frac{576x^8-2304x^7+3824x^6-3408x^5+1585x^4-178x^3-153x^2+58x+9}{144x^{3/2}(x-1)^{3/2}(2x-1)^3(x^2-x-1)}$$

divinding by the leading coefficients we get

$$U^2-\frac{11x^2-3x-1-16x^3+8x^4}{2(x-1)(2x-1)x(x^2-x-1)}\,U$$

$$+\frac{576x^8-2304x^7+3824x^6-3408x^5+1585x^4-178x^3-153x^2+58x+9}{144x^2(x-1)^2(2x-1)^2(x^2-x-1)^2}$$

Denoting u_1,u_2 the roots of $P(U)=0$ we get the two solutions $e^{\int u_1}$ and $e^{\int u_2}$, which are in our case:

$$1/12\,\frac{24\,x^4-48\,x^3+33\,x^2-9\,x-3\pm 2\sqrt{x\,(x-1)\,(2\,x-1)^4}}{x\,(x-1)\,(2\,x-1)\,(x^2-x-1)}$$

The fact that $L^{\circledS m}(y)=0$ was of maximal order for $m=2$ is not a coincidence, but holds for any m when $L(y)$ is a second order equation. This comes from the fact that the linear system given by the first m equations is always in triangular form. To see this note that for $1\le j\le m$ we have by induction that

$$S^{(j)}=\sum_{i=0}^{j}c_{i,j}W_{\underbrace{1,\ldots,1}_{i},0,\ldots,0}$$

where the $c_{i,j}$ are constants and $c_{j,j} \neq 0$. The key to the induction step is that

$$(W_{\underbrace{1,\ldots,1}_{i},0,\ldots,0})' = (i+1)W_{\underbrace{1,\ldots,1}_{i+1},0,\ldots,0} + (m+1-i)aW_{\underbrace{1,\ldots,1}_{i-1},0,\ldots,0}.$$

where the a_i, b_i are integers with $a_i \neq 0$.

The above allows us to write $W_{\underbrace{1,\ldots,1}_{i+1},0,\ldots,0}$ linearly in terms of $(W_{\underbrace{1,\ldots,1}_{i},0,\ldots,0})'$ and $W_{\underbrace{1,\ldots,1}_{i-1},0,\ldots,0}$. This is the Kovacic recurrence of lenght 2 which allows to find all coefficients of $P(U)$ in terms of S and S'. In [7, 18] such a recurrence is given for any m starting from 1 and S'/S.

There is also a recursive way to compute $L^{\circledS m}(y)$ which was found by [1]. With those two recursions, there is no need to solve linear systems for second order equations. However the linear algebra approach gives an easy proof of the recursion and will be helpfull for the presentation of third order equations in the next section.

4 Third order equations

We now consider third order equations $L(y) = y''' + a_1y' + a_0y$. The main difficulties compared to second order equations are

1. $L^{\circledS m}(y)$ is not alway of maximal order. According to [13] Lemma 3.5 the order of $L^{\circledS 3}(y)$ is either 7 if $L^{\circledS 2}(y)$ is not of maximal order, or 9 or 10.

2. Homogeneous forms in three variables over the complex numbers do not always factor into linear forms.

In order to avoid repetitions we define:

$$\sigma_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \sigma_t = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} \omega^3 & 0 & 0 \\ 0 & -\omega^3-1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{and } \sigma_9 = \begin{pmatrix} -\omega^5-\omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & -\omega^5-\omega^2 \end{pmatrix}$$

where $\omega^6+\omega^3+1=0$. And we define the groups $G_{27}=<\sigma_c,\sigma_3>$, $G_{54}=<\sigma_c,\sigma_3,\sigma_t>$, $G_{81}=<\sigma_c,\sigma_9>$ and $G_{162}=<\sigma_c,\sigma_9,\sigma_t>$ whose orders correspond to the indices. We note that for each of the 4 groups all faithful irreducible unimodular characters of degree 3 are conjugate. We also need the 8 finite primitive subgroups of $SL(3,C)$ denoted $A_5^{SL_3}$, $A_5^{SL_3}\times C_3$, $G_{168}^{SL_3}$, $G_{168}^{SL_3}\times C_3$, $A_6^{SL_3}$, $H_{216}^{SL_3}$ and H_{72} defined in [14].

4.1 The general formula

It turns out that the generic case for third order equations is given by $m=3$ and $L^{\circledS 3}(y)=0$ is of order 10. In this case using the same approach as for second order equations we will be able to give a formula for $L^{\circledS 3}(y)=0$ by considering the first 10 derivatives and looking for a nonzero element in the kernel of the transpose of A_{10}. Also the 10 by 10 matrix A_9 obtained from the first 9 derivatives will be invertible and thus if the value s of S and its derivatives are known, we will be able to solve the system in a unique way and thus get the value of the coefficients

$$w_{0,0,0},\, w_{1,0,0},\, w_{1,1,0},\, w_{1,1,1}$$

of $P(U)$. Since the formulas are too big to be included, we therefore only include the matrix A_9 expressed in the basis

$$(W_{0,0,0}, W_{1,0,0}, W_{2,0,0}, W_{1,1,0}, W_{2,1,0}, W_{1,1,1}, W_{2,2,0}, W_{2,1,1}, W_{2,2,1}, W_{2,2,2})$$

Note that Maple contains algorithms to construct symmetric powers (where A_{10} would be needed).

$$\left[\begin{array}{l}
1\\
0\\
0\\
a_0\\
-3a_0'\\
3(a_1a_0-a_0'')\\
-3(a_0'''-3a_1'a_0-a_1a_0'-21a_0^2)\\
3(6a_1''a_0+75a_0a_0'-a_1^2a_0+a_1a_0''+4a_1'a_0'-a_0^{(iv)})\\
3(a_1a_0'''-a_1^2a_0'-114a_1a_0^2+5a_1'a_0''+123a_0a_0''+75a_0'^2+10a_1'''a_0+10a_1''a_0'-8a_1a_0a_1'-a_0^{(v)})\\
3(a_1a_0^{(iv)}-513a_1'a_0^2+a_1^3a_0+189a_0a_0'''-18a_1'^2a_0-a_1^2a_0''-10a_1a_0'a_1'+20a_1'''a_0'+273a_0'a_0''-a_0^{(vi)}+\\
\qquad 15a_1^{(iv)}a_0-432a_1a_0a_0'-21a_1a_0a_1''+6a_1'a_0'''-729a_0^3+15a_1''a_0'')
\end{array}\right.$$

$$\begin{array}{l}
0\\
3\\
0\\
-3a_1\\
3(-9a_0-a_1')\\
3(a_1^2-12a_0'-a_1'')\\
3(4a_1a_1'-15a_0''-a_1'''+54a_1a_0)\\
3(90a_1a_0'-18a_0'''+225a_0^2-a_1^{(iv)}-a_1^3+4a_1'^2+117a_1'a_0+7a_1a_1'')\\
3(219a_1''a_0-9a_1^2a_1'+219a_1'a_0'-21a_0^{(iv)}-243a_1^2a_0-a_1^{(v)}+1011a_0a_0'+141a_1a_0''+15a_1'a_1''+11a_1a_1''')\\
3(1980a_0a_0''+210a_1a_0'''-450a_1^2a_0'+16a_1a_1^{(iv)}+468a_1''a_0'-28a_1a_1'^2+26a_1'a_1'''-1593a_1a_0a_1'-24a_0^{(v)}+\\
\qquad 15a_1''^2-a_1^{(vi)}+a_1^4+1236a_0'^2+375a_1'a_0''-3483a_1a_0^2-22a_1^2a_1''+369a_1'''a_0)
\end{array}$$

$$
\begin{array}{ll}
0 & 0 \\
0 & 0 \\
3 & 6 \\
0 & 0 \\
-3a_1 & -24a_1 \\
3(-21a_0 - 2a_1') & 30(-a_1' - 3a_0) \\
3(-33a_0' + a_1^2 - 3a_1'') & 6(16a_1^2 - 27a_0' - 6a_1'') \\
6(3a_1a_1' - 2a_1''' - 24a_0'' + 57a_1a_0) & 42(30a_1a_0 - 6a_0'' - a_1''' + 9a_1a_1') \\
3(729a_0^2 - 66a_0''' + 10a_1'^2 + 399a_1'a_0 - a_1^3 + 204a_1a_0' + & 6(468a_1a_0' + 112a_1a_1'' + 474a_1'a_0 + 351a_0^2 + 67a_1'^2 - 64a_1^3 \\
\qquad 13a_1a_1'' - 5a_1^{(iv)}) & \qquad -8a_1^{(iv)} - 60a_0''') \\
9(318a_1''a_0 + 1215a_0a_0' + 16a_1'a_1'' + 8a_1a_1''' - 165a_1^2a_0 - & 18(303a_1a_0'' + 387a_1'a_0' + 61a_1a_1''' - 735a_1^2a_0 - 181a_1^2a_1' + \\
\qquad 4a_1^2a_1' - 2a_1^{(v)} + 274a_1'a_0' + 115a_1a_0'' - 29a_0^{(iv)}) & \qquad 711a_0a_0' + 87a_1'a_1'' + 315a_1''a_0 - 27a_0^{(iv)} - 3a_1^{(v)})
\end{array}
$$

$$
\begin{array}{ll}
0 & 0 \\
0 & 0 \\
0 & 0 \\
18 & 6 \\
0 & 0 \\
-90a_1 & -60a_1 \\
162(-3a_0 - a_1') & 90(-a_0 - a_1') \\
126(3a_1^2 - 8a_0' - 2a_1'') & 42(13a_1^2 - 3a_1'' - 6a_0') \\
36(201a_1a_0 + 57a_1a_1' - 10a_1''' - 50a_0'') & 168(14a_1a_1' - 3a_0'' + 12a_1a_0 - a_1''') \\
18(1215a_1'a_0 + 978a_1a_0' + 249a_1a_1'' - 162a_0''' + 162a_1'^2 - & 6(459a_1'^2 - 144a_0''' - 820a_1^3 + 810a_1'a_0 + 756a_1a_1'' + \\
\qquad 85a_1^3 - 27a_1^{(iv)} + 729a_0^2) & \qquad 1224a_1a_0' - 36a_1^{(iv)} + 351a_0^2)
\end{array}
$$

$$
\left.
\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
18 & 36 & 0 & 0 \\
0 & 0 & 90 & 0 \\
-90a_1 & -450a_1 & 0 & 90 \\
252(-3a_0 - a_1') & 126(-6a_0 - 7a_1') & -1260a_1 & 0 \\
126(378a_1^2 - 504a_1'' - 1764a_0') & 504(9a_1^2 - 5a_0' - 3a_1'') & 252(-13a_1' - 9a_0) & -1260a_1 \\
108(102a_1a_0 - 8a_1''' - 33a_0'' + & 108(229a_1a_1' - 22a_1''' - & 378(35a_1^2 - 18a_1'' - 24a_0') & 2268(-2a_1' - a_0) \\
\qquad 26a_1a_1') & \qquad 54a_0'' + 165a_1a_0) & &
\end{array}
\right]
$$

To get the polynomial $P(U)$ for a particular equation $L(y) = 0$ one should replace the values of a_0 and a_1 in the matrix A_9 (and A_{10}) and work from there.

Example 4.1 *Consider the third order equation $L(y)$ given by*

$$
\frac{d^3}{dx^3}y(x) + \frac{123x^2 - 103x + 108}{144x^2\,(x^2 - 2\,x + 1)}\,\frac{d}{dx}y(x) - \frac{748x^3 - 937x^2 - 648 + 1605x}{864x^3\,(x^3 - 3\,x^2 + 3x - 1)}\,y(x)
$$

Factorisation shows that it is an irreducible equation. The equation $L^{\circledS 3}(y) = 0$ is of order 10 and has (up to multiples) a unique exponential solution $x^{7/2} - 2x^{5/2} + x^{3/2}$ where f^2 is rational. Since the semi-invariant must factor into linear forms (cf. [13], Theorem 4.6), our generic formula with $m = 3$ can be applied. Solving the 10×10 system for $s = x^{7/2} - 2x^{5/2} + x^{3/2}$ we get

$$
w_{0,0,0} \;=\; x^{7/2} - 2x^{5/2} + x^{3/2}
$$

$$
\begin{aligned}
w_{1,0,0} &= \frac{7}{6}x^{5/2} - \frac{5}{3}x^{3/2} + \frac{1}{2}x^{1/2} \\
w_{1,1,0} &= \frac{187\,x^2 - 159\,x + 36}{144\,x^{1/2}} \\
w_{1,1,1} &= \frac{1156\,x^3 - 1467\,x^2 + 675\,x - 108}{864\,x^{3/2}(x-1)}
\end{aligned}
$$

This gives us the following minimal polynomial $P(U)$:

$$
U^3 - \frac{7\,x-3}{2x\,(x-1)}\,U^2 + \frac{187\,x^2 - 159\,x + 36}{48\,x^2\,(x-1)^2}\,U - \frac{1156\,x^3 - 1467\,x^2 + 675\,x - 108}{864\,x^3\,(x-1)^3}
$$

4.2 Algorithm for third order equations

The following algorithm either computes all Liouvillian solutions of a given third order differential equation $L(y) = 0$ via the previous formula, or says that the differential Galois group $\mathcal{G}$ is one of the 12 finite groups defined above (in which case procedures exists to compute the algebraic solutions), or shows that $\mathcal{G}$ is isomorphic to PGL_2 or $PGL_2 \times C_3$ (in which case it is still possible to solve the differential equation using second order equations), or that $\mathcal{G}$ is SL_3 (in which case it is not possible to solve the differential equation using second order equations or Liouvillian solutions). The following steps must be performed in the order given:

1. **The reducible case**

Factor $L(y) = y''' + a_1 y' + a_0 y$. If $L(y) = L_1(L_2(y))$ is reducible, then using the reduction method of D'Alembert, we can solve $L(y) = 0$ in terms of second order equations. To find all Liouvillian solutions of $L(y) = 0$ one has to proceed as in [13] and use a factorisation of $L(y)$ and its adjoint.

2. **If $L^{\circledS 3}(y)$ is of order 7**

In this case, $2a_0 = a_1'$. We then set $\tilde{L}(y) = y'' + \frac{a_1(x)}{4} y$ and look for Liouvillian solutions of $\tilde{L}(y) = 0$. If $\tilde{L}(y) = 0$ has two Liouvillian solutions y_1, y_2, then $y_1^2, y_1 y_2, y_2^2$ will be a basis of Liouvillian solutions of $L(y) = 0$. Otherwise $L(y) = 0$ does not have Liouvillian solutions (but can be solved in terms of solutions of $\tilde{L}(y) = 0$).

3. **The imprimitive case**

Compute the exponential solutions of $L^{\circledS 3}(y) = 0$.

(a) If there is a two dimensional space of invariants of degree 3, then $\mathcal{G} \cong G_{27}$.

(b) If there is a two dimensional space of semi-invariants of degree 3 and order 2, then $\mathcal{G} \cong G_{54}$

(c) If there is (up to multiples) 1 invariant of degree 3 and 3 or 2 (if $L^{\circledS 3}(y)$ is of order 9) semi-invariants of degree 3 and order 3, then $\mathcal{G} \cong G_{81}$

(d) If there is (up to multiples) 1 invariant of degree 3 and if either $L^{\circledS 3}(y)$ is of order 9 or has 1 semi-invariant of degree 3 and order 6, then $\mathcal{G} \cong G_{162}$.

(e) **Generic case where the formula holds:** If there is (up to multiples) one invariant f of degree 3 or one semi-invariant f of degree 3 and order 2, then, put $S = f$. The equation $L^{\circledS 3}(y) = 0$ is of order 10 and we find $P(U)$ of degree 3 by solving the linear system where a_0 and a_1 have been replaced by their values.

4. The primitive case

(a) If there is a one or two dimensional space of semi-invariants of degree 3 and order 4, then $\mathcal{G} \cong F_{36}$

(b) Compute the exponential solutions of $L^{\circledS 2}(y) = 0$. If there is a solution which is a cube root of a rational solution, then

  i. If $L^{\circledS 6}(y) = 0$ is of order 27 or has a two dimensional space of rational solutions, then $\mathcal{G} \cong A_5 \times C_3$.

  ii. otherwise $\mathcal{G} \cong PGL(2, \mathcal{C}) \times C_3$

  If there is a rational solution, then

  i. If $L^{\circledS 6}(y) = 0$ is of order 27 or has a two dimensional space of rational solutions, then $\mathcal{G} \cong A_5$.

  ii. otherwise $\mathcal{G} \cong PGL(2, \mathcal{C})$

(c) Compute the exponential solutions of $L^{\circledS 4}(y) = 0$.

  i. If there is a solution which is a cube root of a rational solution, then $\mathcal{G} \cong G_{168} \times C_3$.

  ii. If there is a rational solution, then $\mathcal{G} \cong G_{168}$.

  iii. If $L^{\circledS 4}(y) = 0$ is of order 14 then $\mathcal{G}$ is isomorphic to either G_{168} or $G_{168} \times C_3$. In order to decide which of the two group it is we compute $L^{\circledS 14}(y) = 0$.

A. If there is a solution which is a cube root of a rational solution, then $\mathcal{G} \cong G_{168} \times C_3$.

B. If there is a rational solution, then $\mathcal{G} \cong G_{168}$.

(d) Compute the exponential solutions of $L^{\circledS 6}(y) = 0$.

i. If there are 2 or 3 solutions whose squares are rational (but not multiples of each others), then $\mathcal{G} \cong H_{72}^{SL_3}$.

ii. If there is a rational solution, then $\mathcal{G} \cong A_6^{SL_3}$.

iii. If there is a solution whose cube is rational, then $\mathcal{G} \cong H_{216}^{SL_3}$.

iv. If $L^{\circledS 6}(y)$ is of order 27 then $\mathcal{G}$ is isomorphic to either $H_{216}^{SL_3}$ or $A_6^{SL_3}$. In order to decide which of the two groups it is we compute $L^{\circledS 9}(y)$.

A. If there is a rational solution, then $\mathcal{G} \cong H_{216}^{SL_3}$.

B. otherwise $\mathcal{G} \cong A_6^{SL_3}$.

(e) $\mathcal{G} \cong SL(3, \mathcal{C})$ and the equation has no Liouvillian solution and cannot be solved in terms of solutions of second order equations.

The rest of the paper is a proof of the correctness of the above algorithm (given in the order of the algorithm). We note that it is always possible to solve a linear differential equation whose differential Galois group is a known finite group [3, 14, 17] and that if $\mathcal{G}$ is isomorphic to either PGL_2 or $PGL_2 \times C_3$ it is possible to effectively solve $L(y) = 0$ in terms of second order equations [11]. For the approach in the imprimitive case, the above decision procedure is different from the one given in [6]. We also note that it is easier to just test the existence of Liouvillian solutions (cf. [13]) and that necessary conditions similar to those of the Kovacic algorithm exist (cf. [15]).

4.3 If the third symmetric power is of order 7

Let $L(y) = y''' + a_1 y' + a_0 y$. According to [13] Lemma 3.5 the order of $L^{\circledS 3}(y)$ is either 7 if $L^{\circledS 2}(y)$ is not of maximal order, or 9 or 10. We compute $L^{\circledS 2}(y)$ which is of order 5 or 6. For this we compute the derivatives of $S = Z_1 Z_2$

and get a system

$$\begin{pmatrix} S \\ S' \\ S'' \\ S''' \\ S^{(iv)} \\ S^{(v)} \\ S^{(vi)} \end{pmatrix} = A \begin{pmatrix} W_{0,0} \\ W_{1,0} \\ W_{1,1} \\ W_{2,0} \\ W_{2,1} \\ W_{2,2} \end{pmatrix}$$

where A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ -2a_0 & -a_1 & 0 & 0 & 3 & 0 \\ -2a_0' & -5a_0 - a_1' & -8a_1 & -a_1 & 0 & 6 \\ 2(a_1a_0 - a_0'') & -7a_0' + a_1^2 - a_1'' & -10(a_1' + a_0) & -2a_1' - 11a_0 & -15a_1 & 0 \end{bmatrix}$$

The order of $L^{ⓢ2}(y)$ is 5 if and only if the transposed matrix has a non trivial kernel. Performing a fraction free gaussian elimination on the transposed matrix we get:

$$\begin{bmatrix} 1 & 0 & 0 & -2a_0 & -2a_0' & 2a_1a_0 - 2a_0'' \\ 0 & 1 & 0 & -a_1 & -5a_0 - a_1' & -7a_0' + a_1^2 - a_1'' \\ 0 & 0 & 2 & 0 & -8a_1 & -10a_1' - 10a_0 \\ 0 & 0 & 0 & 6 & 0 & -30a_1 \\ 0 & 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 108(a_1' - 2a_0) \end{bmatrix}$$

which shows that $L^{ⓢ2}(y)$ is of order 5 if and only if $a_1' = 2a_0$. Thus

Proposition 4.2 *Let $L(y) = y''' + a_1y' + a_0y$. Then $L^{ⓢ3}(y)$ is of order 7 if and only if $a_1' = 2a_0$. In this case we have that*

1. *$L(y) = \tilde{L}^{ⓢ2}(y)$ where $\tilde{L}(y) = y'' + \frac{a_1(x)}{4}y$.*

2. *The equation $L(y) = 0$ has Liouvillian solutions if and only $\tilde{L}$ has a basis $(\tilde{y}_1, \tilde{y}_2)$ of Liouvillian solutions. In this case the set $\{\tilde{y}_1^2, \tilde{y}_1\tilde{y}_2, \tilde{y}_2^2\}$ is a fundamental system of solutions of $L(y) = 0$*

Proof. - The above computation shows that the condition is necessary and sufficient (see also [11], Lemma 3.4). Computing the symmetric power of $\tilde{L}^{\circledS 2}(y)$ gives L. The differential Galois group $\tilde{\mathcal{G}}$ of $\tilde{L}(y)$ is either $SL(2,C)$ or $\tilde{\mathcal{G}}^\circ$ is solvable, i.e. $\tilde{L}(y)$ has only Liouvillian solutions (cf. [14]). We denote $\tilde{V}$ the solution space of $\tilde{L}(y)$. If $\tilde{\mathcal{G}} \cong SL(2,C)$, then the reprensentation of $SL(2,C)$ on $V = \mathrm{Sym}(\tilde{V})$ is $PGL(2,C)$ and $L(y) = 0$ has no Liouvillian solutions. Otherwise let $\{\tilde{y}_1, \tilde{y}_2\}$ be a fundamental system of solutions of $\tilde{L}$. Then $\{\tilde{y}_1^2, \tilde{y}_1\tilde{y}_2, \tilde{y}_2^2\}$ is a fundamental system of solutions of $L(y) = 0$ consisting of Liouvillian solutions. Conversely, if $\mathcal{G}^\circ$ is solvable (i.e. all solutions of $L(y) = 0$ are liouvillian) then $\tilde{y}_i^2$ is Liouvillian and thus also $\tilde{y}_i$. $\square$

4.4 The imprimitive case

We consider an imprimitive group $\mathcal{G} \subset SL(3,C)$. This group will, in the basis (Y_1, Y_2, Y_3) corresponding to a system of imprimitivity, contain a diagonal element

$$\sigma_d = \begin{pmatrix} x & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \frac{1}{xz} \end{pmatrix}$$

and, since the group acts transitively on the system of imprimitivity, a cyclic element which we can assume to be of the form

$$\sigma_c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where $x \cdot z \neq 0$. We have a morphism $\varphi: \mathcal{G} \to S_3$ whose image $\mathrm{Im}(\varphi)$ is either A_3 or S_3 (the transitive subgroups of S_3). We may also assume that our element σ_d is non central, i.e. then $x \neq z$. Such a non central diagonal element must exist, since otherwise our group is projectively equivalent to either A_3 or S_3. But the Schur representation groups of A_3 and of S_3 have no irreducible representation of degree 3. In the basis (Y_1, Y_2, Y_3), if $\mathrm{Im}(\varphi) = A_3$ then $Y_1Y_2Y_3$ is an invariant and if $\mathrm{Im}(\varphi) = S_3$ then $Y_1Y_2Y_3$ is a semi-invariant whose square is an invariant.

We are interested in the case where this is the unique (up to constant multiple) semi-invariant of degree 3 and order ≤ 2 that factors into linear forms.

Theorem 4.3 *Let $L(y) = y''' + a_1 y' + a_0 y$ be a third order linear differential equation with unimodular imprimitive Galois group $\mathcal{G}$. Then one of the following holds*

1. *$L^{Ⓢ3}(y)$ has a one dimensional space of rational solutions generated by f. In this case f is the value of an invariant that factors into linear forms.*
2. *$L^{Ⓢ3}(y)$ has a one dimensional space of solution generated by f whose square is a rational function. In this case f is the value of a semi-invariant that factors into linear forms.*
3. *$L^{Ⓢ3}(y)$ has a two dimensional space of rational solutions. In this case $\mathcal{G}(L) \cong G_{27}$.*
4. *$L^{Ⓢ3}(y)$ has no rational solution and a two dimensional space of rational solutions whose square is a rational function. In this case $\mathcal{G}(L) \cong G_{56}$.*

Proof. - We want to find all possible semi-invariants of degree 3 and order ≤ 2

$$\begin{aligned} I &= a_1Y_1^3 + a_2Y_2^3 + a_3Y_3^3 + b_1{Y_1}^2Y_2 + b_2{Y_2}^2Y_1 + b_3Y_1Y_3^2 + c_1Y_2^2Y_3 \\ &\quad + c_2{Y_1}^2Y_3 + c_3Y_2Y_3^3 + c_4Y_1Y_2Y_3 \end{aligned}$$

For such a semi-invariant we must have $\sigma_d(I) = \pm I$ and $\sigma_c(I) = \pm I$. For each possible sign combination we perform the substitution and equate the coefficients to get a polynomial system for the coefficients of I:

1. $\sigma_d(I) = -I$ and $\sigma_c(I) = I$. Taking the numerators (the denominators cannot vanish since it is a power of x and z) we get two sets of equations which together give:

$$\begin{gathered} a_3 - a_1, a_1 - a_2, c_2 - b_2, b_1 - c_1, b_3 - b_1, c_1 - b_3, a_2 - a_3, -c_3, -c_2, \\ b_2(z^2x + 1), a_3(1 + x^3z^3), b_2, a_1(1 + x^3), a_2(1 + z^3), c_1(z + x), c_3, \\ c_3(z^2x^3 + 1), c_2(z + x), b_3(z^2x + 1), b_1(x^2z + 1), 2c_4, 1 - wxz(x - z) \end{gathered}$$

   where the last equation is only to guarantee that x, z are non zero and distinct. Computing a Gröbner basis for a total degree in the variables $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, c_4, x, z, w$ we get that the a_i, c_i and b_i must all be zero and that no non zero polynomial I exists in this case.

2. $\sigma_d(I) = -I$ and $\sigma_c(I) = -I$. Here we also get only the solution $I = 0$.

3. $\sigma_d(I) = I$ and $\sigma_c(I) = -I$. Here we also get only the solution $I = 0$.

4. $\sigma_d(I) = I$ and $\sigma_c(I) = I$. Computing a Gröbner basis for the corresponding system (still adding $1 - wxz(x-z)$) for a lex degree in the variables $w, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, c_4, x, z$ we get that

$$c_3, c_2, c_1, b_3, b_2, b_1, a_3(z-1)(z^2+z+1), a_3(x^2+xz+z^2),$$
$$a_2 - a_3, a_1 - a_3, 1 - wxz(x-z), -a_3(1-3w+2z^2x+1)$$

In particular the a_i are all equal. If they are all zero, then $Y_1Y_2Y_3$ is the only semi-invariant of the type we are looking for. Otherwise we must have $z^3 = 1$ and $x^3 = 1$. Since the determinant of σ_d is one and two of the cube roots in σ_d are distinct, all three cube roots appear in σ_d. Using σ_c we can permute the cube roots in all possible ways. Multiplying them we get all 9 possible combinations of cube roots. Counting cosets of σ_c we get that the group generated by the diagonal and cyclic elements must be a primitive group G_{27}. If $\mathrm{Im}(\varphi) = A_3$ then $\mathcal{G} \cong G_{27}$ otherwise G_{27} is a subgroup of index 2 and thus a normal subgroup of $\mathcal{G}$. Among the 15 non isomorphic groups of order 54 there is a unique group having those properties which is isomorphic to G_{54} .

In order to prove the theorem, it is sufficient to note that the (semi-) invariant that factors into linear forms cannot be zero. $\square$

The above result corresponds to the result in [4, 17] where it is proven that those two groups have four systems of imprimitivity and thus four semi-invariants that factor into linear forms.

Having (up to two exceptions) a unique (value) of a semi-invariant of degree 3 and order ≤ 2, we now want to exclude the possibility that $L^{\circledS 3}(y)$ is of order 9 in order to be able to apply our generic formula from section 4.1.

Proposition 4.4 *Let $L(y) = y''' + a_1y' + a_0y$ be a third order linear differential equation whose differential Galois group $\mathcal{G} \subseteq SL(3, C)$ is imprimitive. Then $L^{\circledS 3}(y)$ is of order 9 if and only if the representation of $\mathcal{G}$ is isomorphic to one of the following*

1. *The group G_{27}. In this case there are (up to multiples) two invariants of degree 3 and 8 semi-invariants of degree 3 and order 3.*

2. *the group G_{81}. In this case there are (up to multiples) 1 invariant of degree 3 and 3 semi-invariants of degree 3 and order 3.*

3. *The group G_{162} In this case there are (up to multiples) 1 invariant of degree 3 and 1 semi-invariant of degree 3 and order 6.*

Proof. - We keep the same notation for I as in the proof of Theorem 4.3. In order for $L^{\circledS 3}(y)$ to be of order 9, a semi-invariant I of degree 3 must evaluate to zero at the solutions of $L(y)=0$. This semi-invariant cannot be the one that factors into linear forms. We thus look for a semi-invariant I such that $\sigma_c(I)=\Lambda_1 \cdot I$ and $\sigma_d(I)=\Lambda_2 \cdot I$. Since Λ_1 and Λ_2 are values of a one dimensional character of $\mathcal{G}$ they must be non zero. We get the following system:

$$\begin{gathered}b_2, a_1-\Lambda_1 a2, c_1-\Lambda_1 b_3, -c_4(-1+\Lambda_1), c_2-\Lambda_1 b_2, -\Lambda_1 b_1+b_3,\\ a_3-\Lambda_1 a_1, b_1(x^2z-\Lambda_2), a_1(x^3-\Lambda_2), -\Lambda_1 c_2, a_2-\Lambda_1 a_3, -\Lambda_1 c_3, -\Lambda_1 c_1+b_1,\\ -a_3(\Lambda_2 x^3 z^3-1), -c_3(\Lambda_2 z^2 x^3-1), -c_1(-z+\Lambda_2 x), a_2(z^3-\Lambda_2), -c_4(\Lambda_2-1),\\ -b_3(\Lambda_2 x z^2-1), b_2(-\Lambda_2+xz^2), c_2(-\Lambda_2 z+x), c3, 1-w(xz)(x-z)\Lambda_1\Lambda_2\end{gathered}$$

Computing a lexicographic Groebner basis with

$$w, \Lambda_1, \Lambda_2, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, c_4, x, z$$

we get

$$\begin{gathered}c_3, c_2, c_1(z^3-1), c_1(x^2+zx+z^2), c_1c_4, b_3(z^3-1), b_3(x^2+zx+z^2),\\ b_3c_4, (b_3-c_1)(b_3^2+c_1b_3+c_1^2), b_2, b_1(z^3-1), b_1(x^2+zx+z^2), c_4b_1,\\ -b_3^2+b_1c_1, b_3b_1-c_1^2, -c_1b_3+b_1^2, a_3(z^9-1), a_3(x^2+zx+z^2), a_3c_4(z^3-1),\\ a_3c_1, b_3a_3, a_3b_1, a_2(z^9-1), a_2(x^2+zx+z^2), c_4(a_2-a_3), c_1a_2, b_3a_2, a_2b_1,\\ (a_2-a_3)(a_2^2+a_2a_3+a_3^2), a_1(z^9-1), a_1(x^2+zx+z^2), c_4(a_1-a_3), c_1a_1, b_3a_1, a_1b_1,\\ a_1a_3-a_2^2, -a_3^2+a_1a_2, a_1^2-a_2a_3, c_4(\Lambda_2-1), c_1(\Lambda_2+xz^2+1), b_3(\Lambda_2+xz^2+1),\\ b_1(\Lambda_2+xz^2+1), -a_3(z^3-\Lambda_2), -a_2(z^3-\Lambda_2), -a_1(z^3-\Lambda_2),\\ c_4(-1+\Lambda_1), \Lambda_1c_1-b_1, -c_1+\Lambda_1b_3, \Lambda_1b_1-b_3, \Lambda_1a_3-a_2,\\ \Lambda_1a_2-a_1, \Lambda_1a_1-a_3, c_4(-1+zx^2w-wxz^2), b_3xz^2+3wc_1+2b_3,\\ z^2xb_1+3wb_3+2b_1, 3wb_1+xz^2c_1+2c_1, 3wa_3-2xa_1z^2-z^3a_1,\\ 3wa_2-2xa_3z^2-a_3z^3, 3wa_1-2xz^2a_2-a_2z^3, 1-w(xz)(x-z)\Lambda_1\Lambda_2\end{gathered}$$

Assuming that $\mathcal{G}$ is neither G_{27} nor G_{54} our new semi-invariant cannot be of order 2 and thus that Λ_1 and Λ_2 cannot be 1 at the same time, which shows that $c_4=0$. If $c_1 \neq 0$, then $z^3=1=x^3=0$ showing that in this case the diagonal subgroup of $\mathcal{G}$ is of order 9. According to the proof of Theorem

4.3 the only possibilities are G_{27} and G_{54}. If $c_1 = 0$, then also $b_1 = b_3 = 0$ so that only the a_i could be non zero. If a_3 is zero than $a_1 = a_2 = 0$ showing that $I = 0$. Thus a semi-invariant of the form we are looking for exists only if $a_3 \neq 0$. In this case $z^9 = 1$. If for all z we have $z^3 = 1$ then again the diagonal subgroup of $\mathcal{G}$ is of order 9 and we get no new group. If $z^6 + z^3 + 1 = 0$, then there is a non scalar diagonal element of order 9 in $\mathcal{G}$ and from $x^2 + xz + z^2$ we get that $x = z^3$ or $x = -1 - z^3$ is a cube root of unity. As in the proof of Theorem 4.3, using σ_c and multiplications we can permute those entries in all possible ways, showing that our group is of order at most $9 \cdot 9 \cdot 3 \cdot 2$ and contained in the group generated by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^3 & 0 \\ 0 & 0 & \omega^5 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $\omega^6 + \omega^3 + 1 = 0$. Among the subgroups having a faithful irreducible representation of degree 3 are groups of order 3^3, 3^4, $3^3 2$, $3^4 2$ and 3^5 which are all isomorphic. If the order is $3^3 2$ the group is G_{54}. In this case there are no semi-invariants and the space of invariants is spanned by invaraints that factor into linear forms and thus cannot have a zero value. Decomposing the character of the third symmetric power we get that only for those of order 3^3, $3^3 2$ and $3^4 2$ we have a semi-invariant whose square is not and invariant. Counting them and comparing the groups gives the result. $\square$

In the following example we prove for G_{162} that $L^{\circledS 3}(y) = 0$ can be of order 9 in the imprimitive case.

Example 4.5 *We construct an equation with Galois group G_{162} using the method of [8]. For this we use the generators g_1 and g_2:*

$$g_1 = \begin{pmatrix} 0 & 0 & -\omega^5 - \omega^2 \\ -\omega^5 - \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $\omega^6 + \omega^3 + 1 = 0$. Taking g_2 as mondodromy at 0, g_1 at 1 and $g_3 = (g_1 g_2)^{-1}$ at ∞ we get respectively as exponents $\{0, \frac{-1}{2}, \frac{1}{2}\}$, $\{\frac{-11}{18}, \frac{-4}{9}, \frac{-17}{18}\}$ and $\{2, \frac{4}{3}, \frac{5}{3}\}$. Since the exponents at 0 differ by an integer, we get the remaining parameter by imposing that no logarithmic term appears. This gives the equation:

$$L_{162} = y''' + \frac{8x - 3}{(x-1)x} y'' + \frac{1536x^2 - 1156x + 81}{108x^2(x-1)^2} y' + \frac{3240x^2 - 3674x + 621}{729(x-1)^3 x^2} y$$

The equation $L_{162}^{\circledS 3}(y) = 0$ *is of order* 10 *and has two exponential solutions (semi-invariants)*

$$f_1 = x^{-\frac{3}{2}}(x-1)^{-\frac{5}{2}} \quad \textit{and} \quad f_2 = x^{-\frac{3}{2}}(x-1)^{-\frac{17}{6}}$$

corresponding to characters of degree 2 *and* 6*. By the above the first is the value of the semi-invariant that factors and the existence of the second proves that the differential Galois group is* G_{162}*. In the basis of the series solutions* $s_1(3) = 1, s_1'(3) = 0, s_1''(3) = 0,\ s_2(3) = 0, s_2'(3) = 1, s_2''(3) = 0$, $s_3(3) = 0, s_3'(3) = 0, s_3''(3) = 1$ *the semi-invariant* S_2*, whose value* f_2*, is*

$$\begin{aligned} &19591041024Y_1^3 - 5036188608Y_2^3 + 6598075315Y_3^3 - 37549495296Y_2Y_1^2 \\ &+41373980928Y_3Y_1^2 + 23869022976Y_1Y_2^2 + 16556079888Y_3Y_2^2 + 28755247536Y_1Y_3^2 \\ &-18115036884Y_2Y_3^2 - 52443490176Y_3Y_2Y_1 \end{aligned}$$

and the semi-invariant S_1*, whose value is* f_1*, is:*

$$\begin{aligned} &19591041024Y_1^3 - 3851219520Y_2^3 + 4236702871Y_3^3 - 34284321792Y_1^2Y_2 \\ &+35629694208Y_1^2Y_3 + 19938721536Y_2^2Y_1 + 11950106256Y_2^2Y_3 + 21408513840Y_3^2Y_1 \\ &-12337172316Y_3^2Y_2 - 41352985728Y_1Y_2Y_3 \end{aligned}$$

and the last one factors into linear forms over C*. In order to show that the third symmetric power of an equation having such a group can be of ordre* 9 *instead of* 10 *it is enough to compute* $\alpha_0, \alpha_1, \alpha_2$ *so that the series* $\tilde{s}_i = s_i + (\sum_{j=0}^{2} \alpha_j x^j) s_i'$ *take the value* 0 *on* S_2 *(via a Gröbner basis it is not difficult to show that such* α_i *exist, but the result is too long to be included). The differential equation* $\tilde{L}_{162}(y) = 0$ *for the new series will be equivalent to* $L_{162}(y)$ *and will also be irreducible of third order with differential Galois group* G_{162} *(cf. [11]). By construction* $\tilde{L}_{162}^{\circledS 3}(y) = 0$ *will be of order* 9*.*

4.5 The primitive case

For a representation $\mathcal{G} \subseteq SL(3,\mathbb{C})$ of a primitive group we denote χ_m the character of the m^{th} symmetric powers of this representation. We look for one dimensional characters in the decomposition χ_m (this will be semi-invariants of $\mathcal{G}$). The trivial character will be denoted $\mathbb{1}$ the trivial character and 1_i a character of order i (if there are several we denote them $1_{i,a}$, $1_{i,b}, \ldots$ and if there appear twice we denote them $2 \cdot 1_i$). We did not include $SL(3,\mathbb{C})$ which acts irreducible on all symmetric powers ([13]).

The following table describes the one dimensional characters that appear in the decomposition of the character of the action of $\mathcal{G}$

m	PSL_2	PSL_2 $\times C_3$	A_5	A_5 $\times C_3$	G_{168}	G_{168} $\times C_3$	$F_{36}^{SL_3}$	$A_6^{SL_3}$	$H_{216}^{SL_3}$	$H_{72}^{SL_3}$
2	$\mathbf{1}$	1_3	$\mathbf{1}$	$1_{3,a}$						
3							$1_{4,a}+1_{4,b}$			
4	$\mathbf{1}$	1_3	$\mathbf{1}$	$1_{3,b}$	$\mathbf{1}$	1_3				
6	$\mathbf{1}$	$\mathbf{1}$	$2\cdot\mathbf{1}$	$2\cdot\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$2\cdot\mathbf{1}+2\cdot 1_2$	$\mathbf{1}$	1_3	$\mathbf{1}+1_{2,a}$ $+1_{2,b}+1_{2,c}$
9							$\mathbf{1}+3\cdot 1_{4,a}$ $+3\cdot 1_{4,b}$		$\mathbf{1}$	$\mathbf{1}$

Note that in the primitive case, the second symmetric power can no longer be of order 5 (since we have already excluded this case). This will allow us to distinguish between A_5 and $A_5\times C_3$ and between PSL_2 and $PSL_2\times C_3$ once we know that it is one of those two groups. The proof of the algorithm now goes by inspection of the table, taking into account that some symmetric power could be of lower order. Note that the first time that the order drops, it can only drop by one. So the only problems left are

1. when χ_6 is of order one less and that we cannot distinguish $A_6^{SL_3}$ and $H_{216}^{SL_3}$. Here we note that the invariant of degree 9 of $H_{216}^{SL_3}$ cannot be zero, since it factors into linear forms.

2. when χ_4 is of order one less and that we cannot distinguish G_{168} and $G_{168}\times C_3$. Working in the basis of [14], the semi-invariant of degree 4 of $G_{168}\times C_3$ (which is an invariant of degree 4 of G_{168}) is $S_4 = Y_1^3Y_3+Y_1Y_2^3+Y_2Y_3^3$ and the semi-invariant of degree 14 is

$$\begin{aligned}
S_{14} \;=\;& Y_1^{14}+\frac{104}{9}Y_1^{11}Y_2Y_3^2-\frac{1430}{9}Y_1^9Y_2^4Y_3+\frac{715}{3}Y_1^8Y_2^2Y_3^4+\frac{572}{9}Y_1^7Y_2^7\\
&+\frac{572}{9}Y_1^7Y_3^7-\frac{4004}{9}Y_1^6Y_2^5Y_3^3-\frac{4004}{9}Y_1^5Y_2^3Y_3^6+\frac{715}{3}Y_1^4Y_2^8Y_3^2\\
&-\frac{1430}{9}Y_1^4Y_2Y_3^9-\frac{4004}{9}Y_1^3Y_2^6Y_3^5+\frac{104}{9}Y_1^2Y_2^{11}Y_3+\frac{715}{3}Y_1^2Y_2^4Y_3^8\\
&-\frac{1430}{9}Y_1Y_2^9Y_3^4+\frac{104}{9}Y_1Y_2^2Y_3^{11}+Y_2^{14}+\frac{572}{9}Y_2^7Y_3^7+Y_3^{14}
\end{aligned}$$

and the resultant of S_4 and S_{14} in the variable Y_3 is:

$$Y_2^{56} + Y_1^{56} - 11922444Y_2^{49}Y_1^7 + 965840274Y_2^{42}Y_1^{14} + 1189620656Y_2^{35}Y_1^{21} + 51908Y_1^{49}Y_2^7 + 1214420963Y_2^{28}Y_1^{28} + 236710152Y_1^{35}Y_2^{21} + 12929770Y_1^{42}Y_2^{14}$$

Thus the vanishing of both semi-invariants which would give a linear relation between Y_1 and Y_2 and thus a contradiction.

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